

# A SHARP BALIAN-LOW UNCERTAINTY PRINCIPLE FOR SHIFT-INVARIANT SPACES

DOUGLAS P. HARDIN, MICHAEL C. NORTINGTON V, AND ALEXANDER M. POWELL

**ABSTRACT.** A sharp version of the Balian-Low theorem is proven for the generators of finitely generated shift-invariant spaces. If generators  $\{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$  are translated along a lattice to form a frame or Riesz basis for a shift-invariant space  $V$ , and if  $V$  has extra invariance by a suitable finer lattice, then one of the generators  $f_k$  must satisfy  $\int_{\mathbb{R}^d} |x| |f_k(x)|^2 dx = \infty$ , namely,  $\widehat{f_k} \notin H^{1/2}(\mathbb{R}^d)$ . Similar results are proven for frames of translates that are not Riesz bases without the assumption of extra lattice invariance. The best previously existing results in the literature give a notably weaker conclusion using the Sobolev space  $H^{d/2+\epsilon}(\mathbb{R}^d)$ ; our results provide an absolutely sharp improvement with  $H^{1/2}(\mathbb{R}^d)$ . Our results are sharp in the sense that  $H^{1/2}(\mathbb{R}^d)$  cannot be replaced by  $H^s(\mathbb{R}^d)$  for any  $s < 1/2$ .

## 1. INTRODUCTION

The uncertainty principle in harmonic analysis is a class of results which constrains how well-localized a function  $f$  and its Fourier transform  $\widehat{f}$  can be. A classical expression of the uncertainty principle is given by the  $d$ -dimensional Heisenberg inequality

$$\forall f \in L^2(\mathbb{R}^d), \quad \left( \int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx \right) \left( \int_{\mathbb{R}^d} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi \right) \geq \frac{d^2}{16\pi^2} \|f\|_{L^2(\mathbb{R}^d)}^4, \quad (1.1)$$

where the Fourier transform  $\widehat{f} \in L^2(\mathbb{R}^d)$  is defined using  $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$ . For background on this and other uncertainty principles, see [26, 32].

There exist versions of the uncertainty principle which not only constrain time and frequency localization of an individual function as in (1.1), but instead constrain the collective time and frequency localization of orthonormal bases and other structured spanning systems such as frames and Riesz bases. A collection  $\{h_n\}_{n=1}^\infty$  in a Hilbert space  $\mathcal{H}$  is a *frame* for  $\mathcal{H}$  if there exist constants  $0 < A \leq B < \infty$  such that

$$\forall h \in \mathcal{H}, \quad A \|h\|_{\mathcal{H}}^2 \leq \sum_{n=1}^{\infty} |\langle h, h_n \rangle_{\mathcal{H}}|^2 \leq B \|h\|_{\mathcal{H}}^2.$$

---

2010 *Mathematics Subject Classification.* Primary 42C15.

*Key words and phrases.* Balian-Low theorem, shift-invariant space, uncertainty principle.

D. Hardin was partially supported by NSF DMS Grant 1109266, NSF DMS Grant 0934630, NSF DMS Grant 1521749, NSF DMS Grant 1516400, and NSF DMS Grant 1412428.

M. Northington was partially supported by NSF DMS Grant 0934630, NSF DMS Grant 1521749, and NSF DMS Grant 1211687.

A. Powell was partially supported by NSF DMS Grant 1211687 and NSF DMS Grant 1521749.

The collection  $\{h_n\}_{n=1}^\infty$  is a *Riesz basis* for  $\mathcal{H}$  if it is a minimal frame for  $\mathcal{H}$ , i.e.,  $\{h_n\}_{n=1}^\infty$  is a frame for  $\mathcal{H}$  but  $\{h_n\}_{n=1}^\infty \setminus \{h_N\}$  is not a frame for  $\mathcal{H}$  for any  $N \geq 1$ . Equivalently,  $\{h_n\}_{n=1}^\infty$  is a Riesz basis for  $\mathcal{H}$  if and only if  $\{h_n\}_{n=1}^\infty$  is the image of an orthonormal basis under a bounded invertible operator from  $\mathcal{H}$  to  $\mathcal{H}$ . Every orthonormal basis is automatically a Riesz basis and a frame, but there exist frames that are not Riesz bases, and Riesz bases that are not orthonormal bases. See [22] for background on frames and Riesz bases.

The following beautiful example of an uncertainty principle for Riesz bases was proven in [30]. If  $\{f_n\}_{n=1}^\infty \subset L^2(\mathbb{R}^d)$  satisfies

$$\sup_n \left( \int_{\mathbb{R}^d} |x - a_n|^{2d+\epsilon} |f_n(x)|^2 dx \right) \left( \int_{\mathbb{R}^d} |\xi - b_n|^{2d+\epsilon} |\widehat{f_n}(\xi)|^2 d\xi \right) < \infty, \quad (1.2)$$

for some  $\epsilon > 0$  and  $\{(a_n, b_n)\}_{n=1}^\infty \subset \mathbb{R}^2$ , then  $\{f_n\}_{n=1}^\infty$  cannot be a Riesz basis for  $L^2(\mathbb{R}^d)$ . Moreover, this result is sharp in that  $\epsilon$  cannot be taken to be zero, see [17, 30].

There has been particular interest in uncertainty principles for bases that are endowed with an underlying group structure. The Balian-Low theorem for Gabor systems is a celebrated result of this type. Given  $f \in L^2(\mathbb{R})$  the associated *Gabor system*  $\mathcal{G}(f, 1, 1) = \{f_{m,n}\}_{m,n \in \mathbb{Z}}$  is defined by  $f_{m,n}(x) = e^{2\pi i m x} f(x - n)$ . The following nonsymmetric version of the Balian-Low theorem states that if  $\mathcal{G}(f, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$  then  $f$  must be poorly localized in either time or frequency.

**Theorem 1.1** (Balian-Low theorems). *Let  $f \in L^2(\mathbb{R})$  and suppose that  $\mathcal{G}(f, 1, 1)$  is a Riesz basis for  $L^2(\mathbb{R})$ .*

(1) *If  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then*

$$\left( \int_{\mathbb{R}} |x|^p |f(x)|^2 dx \right) \left( \int_{\mathbb{R}} |\xi|^q |\widehat{f}(\xi)|^2 d\xi \right) = \infty.$$

(2) *If  $\widehat{f}$  is compactly supported, then*

$$\int_{\mathbb{R}} |x| |f(x)|^2 dx = \infty.$$

*The same result holds with the roles of  $f$  and  $\widehat{f}$  interchanged.*

The original Balian-Low theorem [6, 41] formulated the case  $p = q = 2$  in part (1) of Theorem 1.1 for orthonormal bases. The non-symmetrically weighted  $(p, q)$  versions with  $p \neq q$  in Theorem 1.1 were subsequently proven in [28]. There are numerous extensions of the Balian-Low theorem, e.g., see the surveys [13, 23] and articles [4, 5, 8, 9, 10, 11, 12, 24, 27, 31, 34, 35, 39, 40, 42, 43].

**Overview and main results.** In this paper we will focus on the interesting recent extensions [2, 45] of the Balian-Low theorem to the setting of shift-invariant spaces. Our main goal is to *prove sharp versions of Balian-Low type theorems in shift-invariant spaces*.

Let us begin by recalling some notation on shift-invariant spaces. Given  $f \in L^2(\mathbb{R}^d)$  and  $\lambda \in \mathbb{R}^d$  the translation operator  $T_\lambda : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  is defined by  $T_\lambda f(x) = f(x - \lambda)$ .

**Definition 1.2.** Let  $\Lambda, \Gamma$  be lattices in  $\mathbb{R}^d$  with  $\Lambda \subset \Gamma$ . Fix a  $K$ -tuple  $F = (f_1, \dots, f_K)$  where each  $f_k \in L^2(\mathbb{R}^d)$ . With slight abuse of notation, we denote this by  $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ . Assume at least one  $f_k$  satisfies  $\|f_k\|_2 \neq 0$ , i.e.,  $F$  is nontrivial.

- (1)  $\mathcal{T}^\Lambda(F)$  denotes the system of translations  $\{T_\lambda f : f \in F \text{ and } \lambda \in \Lambda\}$  viewed as a multiset.
- (2)  $V^\Lambda(F) = V^\Lambda(f_1, \dots, f_K)$  denotes the closed linear span of  $\mathcal{T}^\Lambda(F)$  in  $L^2(\mathbb{R}^d)$ . The space  $V^\Lambda(F)$  is said to be a finitely generated shift-invariant space generated by  $F$ . We shall call the elements of  $F$  generators of  $V^\Lambda(F)$ .
- (3) If  $F = \{f\}$  consists of a single function, then  $V^\Lambda(F) = V^\Lambda(f)$  is said to be a singly generated (or principal) shift-invariant space with generator  $f$ .
- (4) The minimal number of generators  $\rho(F, \Lambda)$  of the space  $V^\Lambda(F)$  is defined by
$$\rho(F, \Lambda) = \min\{N \in \mathbb{N} : \exists \text{ an } N\text{-tuple } G = \{g_n\}_{n=1}^N \text{ such that } V^\Lambda(G) = V^\Lambda(F)\}.$$
- (5)  $V^\Lambda(F)$  is said to be  $\Gamma$ -invariant if  $f \in V^\Lambda(F)$  implies that  $T_\gamma f \in V^\Lambda(F)$  for all  $\gamma \in \Gamma$ .
- (6)  $V^\Lambda(F)$  is said to be translation invariant if  $f \in V^\Lambda(F)$  implies that  $T_t f \in V^\Lambda(F)$  for all  $t \in \mathbb{R}^d$ .

In contrast with the Balian-Low theorem for Gabor systems, it is possible for  $f \in L^2(\mathbb{R}^d)$  to be well-localized in both time and frequency and for the system of shifts  $\mathcal{T}^\Lambda(f)$  to be a Riesz basis for  $V^\Lambda(f)$ . For example, if  $f \in C^\infty(\mathbb{R})$  is compactly supported in  $[-1/2, 1/2]$ , then  $\mathcal{T}^\mathbb{Z}(f)$  is an orthonormal basis for  $V^\mathbb{Z}(f)$ . In view of this, a Balian-Low type theorem will not hold for shift-invariant spaces unless extra assumptions on the space are considered.

Our first main result is the following. This result resolves a question posed in [45] concerning the sharp scale of Sobolev spaces needed for Balian-Low type theorems in shift-invariant spaces.

**Theorem 1.3.** Fix lattices  $\Lambda, \Gamma \subset \mathbb{R}^d$  with  $\Lambda \subset \Gamma$  and  $[\Gamma : \Lambda] > 1$ . Suppose that  $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$  is nontrivial and that  $\mathcal{T}^\Lambda(F)$  is a frame for  $V^\Lambda(F)$ . If  $[\Gamma : \Lambda]$  is not a divisor of  $\rho(F, \Lambda)$  and  $V^\Lambda(F)$  is  $\Gamma$ -invariant, then

$$\exists 1 \leq k \leq K \text{ such that } \int_{\mathbb{R}^d} |x| |f_k(x)|^2 dx = \infty.$$

In other words, at least one of the generators satisfies  $\widehat{f_k} \notin H^{1/2}(\mathbb{R}^d)$ .

Here,  $[\Gamma : \Lambda]$  denotes the index of the lattice  $\Lambda$  in  $\Gamma$ , see Section 2.1. For singly generated shift-invariant spaces, Theorem 1.3 takes the following form.

**Corollary 1.4.** Fix lattices  $\Lambda, \Gamma \subset \mathbb{R}^d$  with  $\Lambda \subset \Gamma$  and  $[\Gamma : \Lambda] > 1$ . Suppose  $f \in L^2(\mathbb{R}^d)$ ,  $\|f\|_2 \neq 0$ , and  $\mathcal{T}^\Lambda(f)$  forms a frame for  $V^\Lambda(f)$ . If  $V^\Lambda(f)$  is  $\Gamma$ -invariant, then  $\int_{\mathbb{R}^d} |x| |f(x)|^2 dx = \infty$ .

To put Theorem 1.3 in perspective, note that all previously existing results in the literature, see [2, 45], either give a weaker conclusion or require stronger hypotheses. In particular, the foundational Theorem 1.2 in [2] addresses singly generated shift-invariant spaces in dimension  $d = 1$  and gives the weaker conclusion that the generator  $f \in L^2(\mathbb{R})$  satisfies  $\widehat{f} \notin H^{1/2+\epsilon}(\mathbb{R})$  whenever  $\epsilon > 0$ . The situation is more extreme in higher dimensions  $d \geq 1$ , where Theorem 1.3 in [45] gives the weaker conclusion that at least one generator satisfies  $\widehat{f_k} \notin H^{d/2+\epsilon}(\mathbb{R}^d)$ .

On the other hand, Theorem 1.2 in [45] shows if the hypothesis of  $\Gamma$ -invariance is replaced by the notably stronger hypothesis of translation invariance, then at least one generator satisfies  $\widehat{f}_k \notin H^{1/2}(\mathbb{R}^d)$ .

Theorem 1.3 is sharp in the sense that  $H^{1/2}(\mathbb{R}^d)$  cannot be replaced by  $H^s(\mathbb{R}^d)$  when  $s < 1/2$ . For example, if  $\chi_I$  is the characteristic function of the set  $I = [-1/2, 1/2]^d$  and  $f(x) = \widehat{\chi_I}(x)$ , then the space  $V^{\mathbb{Z}^d}(f)$  is translation invariant and  $\widehat{f} \in H^s(\mathbb{R}^d)$  for all  $0 < s < 1/2$ , cf. Proposition 1.5 in [2] and Proposition 1.5 in [45].

Theorem 1.3 is precise in the sense that it is possible for only one generator in a multiply generated system to suffer from the localization constraint  $f_k \notin H^{1/2}(\mathbb{R}^d)$ . In particular, we construct examples of  $F = \{f_k\}_{k=1}^K$  that satisfy the hypotheses of Theorem 1.3, and where  $f_K \notin H^{1/2}(\mathbb{R}^d)$  but all other generators  $f_1, \dots, f_{K-1}$  are in  $H^{1/2}(\mathbb{R}^d)$ . This answers a question posed in [45] about the proportion of generators with good localization. See Examples 5.1 and 5.2 in Section 5 for details.

Note that Theorem 1.3 does not contain the compact support hypothesis that is needed in part (2) of Theorem 1.1. For perspective, Theorem 1.1 requires the condition that  $\mathcal{G}(f, 1, 1)$  is a Riesz basis for the entire space  $L^2(\mathbb{R})$ , whereas Theorem 1.3 only requires the weaker assumption that  $\{f(x - n) : n \in \mathbb{Z}\}$  is a frame for its closed linear span  $V^{\mathbb{Z}}(f)$  in  $L^2(\mathbb{R})$ . Moreover, it is known that  $\{f(x - n) : n \in \mathbb{Z}\}$  cannot be a frame for the entire space  $L^2(\mathbb{R})$ , e.g., see the literature on Gabor density theorems, [33].

Our second main result is the following.

**Theorem 1.5.** *Fix a lattice  $\Lambda \subset \mathbb{R}^d$ . Suppose that  $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$  is nontrivial and that  $\mathcal{T}^\Lambda(F)$  is a frame for  $V^\Lambda(F)$ , but is not a Riesz basis for  $V^\Lambda(F)$ . If  $K = \rho(F, \Lambda)$  then*

$$\exists 1 \leq k \leq K \text{ such that } \int_{\mathbb{R}^d} |x| |f_k(x)|^2 dx = \infty.$$

For singly generated shift-invariant spaces, Theorem 1.5 takes the following form.

**Corollary 1.6.** *Fix a lattice  $\Lambda \subset \mathbb{R}^d$ . Suppose  $f \in L^2(\mathbb{R}^d)$  with  $\|f\|_2 \neq 0$ . If  $\mathcal{T}^\Lambda(f)$  is a frame for  $V^\Lambda(f)$ , but is not a Riesz basis for  $V^\Lambda(f)$ , then  $\int_{\mathbb{R}^d} |x| |f(x)|^2 dx = \infty$ , i.e.,  $\widehat{f} \notin H^{1/2}(\mathbb{R}^d)$ .*

Corollary 1.6 stated with the weaker conclusion  $\widehat{f} \notin H^{d/2+\epsilon}(\mathbb{R}^d)$  (or more generally that  $f$  is not integrable) may be considered folklore [29]. The conclusion of Corollary 1.6 with the condition  $\widehat{f} \notin H^{1/2}(\mathbb{R}^d)$  provides a significant and sharp improvement of this.

Theorem 1.5 is closely related to the work in [27]. Note that, unlike Theorem 1.3, Theorem 1.5 does not require an extra lattice invariance assumption for  $V^\Lambda(F)$ . This result is sharp, as can be seen by considering  $V^{\mathbb{Z}^d}(f)$  with  $f(x) = \widehat{\chi_J}(x)$  and  $J = [0, 1/2]^d$ , cf. (2.2). Moreover, Example 5.3 shows that it is possible for only a single generator in Theorem 1.5 to have poor localization.

The remainder of the paper is organized as follows. Section 2 contains background on lattices and shift-invariant spaces; Section 3 contains background on Fourier coefficients and Sobolev spaces. Section 4 contains the proofs of our two main results, Theorems 1.3 and 1.5. In particular, Section 4.1 proves a necessary Sobolev-type embedding for bracket products, Section 4.2 proves a crucial rank property of  $H^{1/2}$ -valued matrices, and Section 4.3 combines

the various preparatory results to prove Theorem 1.3 and Theorem 1.5. Section 5 provides examples related to the main theorems. The Appendix includes the proof of a background lemma concerning Sobolev spaces on the torus.

## 2. SHIFT-INVARIANT SPACES: RIESZ BASES, FRAMES, EXTRA INVARIANCE

In this section we recall necessary background and notation on lattices and shift-invariant spaces.

**2.1. Lattices.** A set  $\Gamma \subset \mathbb{R}^d$  is a (full-rank) lattice if there exists a  $d \times d$  nonsingular matrix  $A$  such that  $\Gamma = A(\mathbb{Z}^d)$ . Equivalently, if the columns of  $A$  are denoted by  $\{a_j\}_{j=1}^d$ , then  $\Gamma = \{\sum_{j=1}^d z_j a_j : z_j \in \mathbb{Z}\}$ . In other words,  $\{a_j\}_{j=1}^d$  is a basis (over  $\mathbb{Z}$ ) for  $\Gamma$ . The dual lattice associated to  $\Gamma$  is defined as  $\Gamma^* = \{\xi \in \mathbb{R}^d : \forall x \in \Gamma, e^{2\pi i x \cdot \xi} = 1\}$ . In terms of the matrix  $A$ , the dual lattice can equivalently be defined as  $\Gamma^* = (A^*)^{-1}(\mathbb{Z}^d)$ .

Let  $\sim$  be the equivalence relation on  $\mathbb{R}^d$  defined by  $x \sim y \iff x - y \in \Gamma$ . We shall say that a set  $S \subset \mathbb{R}^d$  is a fundamental domain of  $\Gamma$  if  $S$  contains precisely one representative of every equivalence class for the relation  $\sim$ . Define  $M_\Gamma \subset \mathbb{R}^d$  by  $M_\Gamma = \{Ax : x \in [-1/2, 1/2)^d\}$ , and note that  $M_\Gamma$  is a fundamental domain of  $\Gamma$ .

Given nested lattices  $\Lambda \subset \Gamma$ , the index of  $\Lambda$  in  $\Gamma$  is denoted by  $[\Gamma : \Lambda]$ , and is defined as the order of the quotient group  $\Gamma/\Lambda$  when  $\Gamma$  and  $\Lambda$  are viewed as discrete subgroups of  $\mathbb{R}^d$ . Moreover,  $[\Gamma : \Lambda] > 1$  if and only if the inclusion  $\Lambda \subset \Gamma$  is strict, i.e.,  $\Lambda \subsetneq \Gamma$ .

A function  $f$  defined on  $\mathbb{R}^d$  will be said to be  $\Gamma$ -periodic if  $f(x + \gamma) = f(x)$  for all  $x \in \mathbb{R}^d$  and  $\gamma \in \Gamma$ .  $L^2(\mathbb{R}^d/\Gamma)$  consists of the  $\Gamma$ -periodic square-integrable functions. Since  $M_\Gamma$  can be identified with the torus  $\mathbb{R}^d/\Gamma$ , the space  $L^2(\mathbb{R}^d/\Gamma)$  consists of  $\Gamma$ -periodic extensions to  $\mathbb{R}^d$  of  $L^2(M_\Gamma)$ .

**2.2. Riesz bases, frames, and shift-invariant spaces.** The question of when a system of translates forms a frame or Riesz basis for a shift-invariance space has been well-studied. Given  $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ , we shall, with slight abuse of notation, let  $F(x)$  denote the  $d \times 1$  column vector whose entries are  $f_k(x)$ ,  $1 \leq k \leq K$ . For any  $x$ ,  $F(x)F^*(x)$  is a  $K \times K$  Hermitian positive semi-definite matrix. Given a lattice  $\Lambda$ , we define the  $\Lambda$ -Gramian of  $F$  to be

$$P_\Lambda(F)(x) = \sum_{\lambda \in \Lambda} F(x - \lambda)F^*(x - \lambda).$$

Note that  $P_\Lambda(F)$  is  $\Lambda$ -periodic and is Hermitian positive semi-definite. Also note that in the case when  $F = \{f\}$  is a singleton,  $P_\Lambda(f) = \sum_{\lambda \in \Lambda} |f(x - \lambda)|^2$ .

Given  $F = \{f_k\}_{k=1}^K$ , let  $\widehat{F} = \{\widehat{f_k}\}_{k=1}^K$ . It is known [14, 25] that  $\mathcal{T}^\Lambda(F)$  forms a Riesz basis for  $V^\Lambda(F)$  if and only if there exists  $t \geq 1$  such that

$$t^{-1}I \leq P_{\Lambda^*}(\widehat{F})(x) \leq tI \quad \text{a.e. } x \in M_{\Lambda^*}. \quad (2.1)$$

Moreover, see [19, 14],  $\mathcal{T}^\Lambda(F)$  forms a frame for  $V^\Lambda(F)$  if and only if there exists  $t \geq 1$  such that

$$t^{-1}P_{\Lambda^*}(\widehat{F})(x) \leq (P_{\Lambda^*}(\widehat{F})(x))^2 \leq tP_{\Lambda^*}(\widehat{F})(x) \quad \text{a.e. } x \in M_{\Lambda^*}. \quad (2.2)$$

The following result addresses the minimal number of generators of shift-invariant spaces, see Proposition 4.1 in [45].

**Proposition 2.1.** *Let  $\Lambda$  be a lattice in  $\mathbb{R}^d$ , and  $F \subset L^2(\mathbb{R}^d)$ . The minimal number of generators of  $V^\Lambda(F)$  is given by*

$$\rho(F, \Lambda) = \text{ess sup}_{x \in \mathbb{R}^d} \left( \text{rank} \left[ P_{\Lambda^*}(\widehat{F})(x) \right] \right).$$

The following theorems address properties of shift-invariant spaces  $V^\Lambda(F)$  that are invariant under a lattice  $\Gamma$  that is larger than  $\Lambda$ , see Theorem 2.1 and Theorem 3.2 in [45] and similar results in [1] and [3]. For simplicity, we state the next two results for lattices  $\Lambda, \Gamma$ , but both results remain true when  $\Lambda \subset \Gamma$  are closed cocompact subgroups of  $\mathbb{R}^d$ .

**Theorem 2.2.** *Let  $\Lambda, \Gamma \subset \mathbb{R}^d$  be lattices with  $\Lambda \subset \Gamma$ . Let  $R \subset \Gamma^*$  be a collection of representatives of the quotient  $\Lambda^*/\Gamma^*$  so that*

$$P_{\Lambda^*}(\widehat{F})(x) = \sum_{k \in R} P_{\Gamma^*}(\widehat{F})(x + k), \text{ a.e. } x \in \mathbb{R}^d.$$

*The space  $V^\Lambda(F)$  is  $\Gamma$ -invariant if and only if*

$$\text{rank} \left[ P_{\Lambda^*}(\widehat{F})(x) \right] = \sum_{k \in R} \text{rank} \left[ P_{\Gamma^*}(\widehat{F})(x + k) \right], \text{ a.e. } x \in \mathbb{R}^d.$$

**Theorem 2.3.** *Let  $\Lambda, \Gamma \subset \mathbb{R}^d$  be lattices with  $\Lambda \subset \Gamma$ . Suppose the space  $V^\Lambda(F)$  is  $\Gamma$ -invariant and  $\mathcal{T}^\Lambda(F)$  forms a frame for  $V^\Lambda(F)$ . Then  $\mathcal{T}^\Gamma(F)$  also forms a frame for  $V^\Lambda(F) = V^\Gamma(F)$ . That is, there exists  $t \geq 1$  such that for a.e.  $x \in \mathbb{R}^d$ ,*

$$t^{-1} P_{\Gamma^*}(\widehat{F})(x) \leq (P_{\Gamma^*}(\widehat{F})(x))^2 \leq t P_{\Gamma^*}(\widehat{F})(x).$$

### 3. BACKGROUND: FOURIER COEFFICIENTS AND SOBOLEV SPACES

In this section we collect necessary background results and notation on Fourier coefficients and Sobolev spaces.

**3.1. Fourier coefficients.** Recall that the Fourier coefficients of  $f \in L^2(\mathbb{R}^d/\Gamma)$  are defined by

$$\forall \xi \in \Gamma^*, \quad \widehat{f}(\xi) = \int_{\mathbb{R}^d/\Gamma} f(x) e^{-2\pi i x \cdot \xi} dh(x) = \frac{1}{|M_\Gamma|} \int_{M_\Gamma} f(x) e^{-2\pi i x \cdot \xi} dx,$$

where  $dx$  is Lebesgue measure and  $dh$  is normalized Haar measure on the compact group  $\mathbb{R}^d/\Gamma$ . Also recall Parseval's theorem

$$\int_{\mathbb{R}^d/\Gamma} |f(x)|^2 dh(x) = \frac{1}{|M_\Gamma|} \int_{M_\Gamma} |f(x)|^2 dx = \sum_{\xi \in \Gamma^*} |\widehat{f}(\xi)|^2, \quad (3.1)$$

and the translation property

$$\forall y \in \mathbb{R}^d, \forall \xi \in \Gamma^*, \quad \widehat{T_y f}(\xi) = \widehat{f}(\xi) e^{-2\pi i y \cdot \xi}. \quad (3.2)$$



**3.2. Sobolev spaces.** Given  $s > 0$ , the Sobolev space  $H^s(\mathbb{R}^d)$  consists of all measurable functions  $f$  defined on  $\mathbb{R}^d$  such that  $\|f\|_{H^s(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty$ . Equivalently,  $f \in H^s(\mathbb{R}^d)$  if and only if  $f \in L^2(\mathbb{R}^d)$  and

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty. \quad (3.3)$$

Recall the following equivalent characterization of (3.3) when  $0 < s < 1$ , e.g., [44],

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)}^2 = C(d, s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} dx dy. \quad (3.4)$$

**3.3. Sobolev spaces of periodic functions.** We shall also need some background on Sobolev spaces of periodic functions.

**Definition 3.1** (Sobolev spaces on the torus). Let  $\Gamma \subset \mathbb{R}^d$  be a lattice with dual lattice  $\Gamma^* \subset \mathbb{R}^d$ . Given  $s > 0$ , define the Sobolev space  $H^s(\mathbb{R}^d/\Gamma) = \{f \in L^2(\mathbb{R}^d/\Gamma) : \|f\|_{\dot{H}^s(\mathbb{R}^d/\Gamma)} < \infty\}$ , where  $\|f\|_{\dot{H}^s(\mathbb{R}^d/\Gamma)} = \left( \sum_{\xi \in \Gamma^*} |\xi|^{2s} |\widehat{f}(\xi)|^2 \right)^{1/2}$ .

The following proposition gives a useful equivalent characterization of  $\|f\|_{\dot{H}^s(\mathbb{R}^d/\Gamma)}$  for  $0 < s < 1$ . Equation (3.5) is a version for  $H^s(\mathbb{R}^d/\Gamma)$  of Proposition 1.3 in [15], and equation (3.6) is an extension to  $H^s(\mathbb{R}^d/\Gamma)$  of the equivalence on page 66 in [18]. We use the notation  $X \asymp Y$  to indicate that there exist absolute constants  $0 < C_1 \leq C_2$  such that  $C_1 X \leq Y \leq C_2 X$ .

**Lemma 3.2.** Fix  $0 < s < 1$ , let  $\Gamma \subset \mathbb{R}^d$  be lattice, and suppose that  $f \in L^2(\mathbb{R}^d/\Gamma)$ . Then

$$\|f\|_{\dot{H}^s(\mathbb{R}^d/\Gamma)}^2 \asymp \int_{M_\Gamma} \int_{M_\Gamma} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} dx dy. \quad (3.5)$$

Moreover, if  $\{a_j\}_{j=1}^d \subset \Gamma$  is a basis for  $\Gamma$  then

$$\|f\|_{\dot{H}^s(\mathbb{R}^d/\Gamma)}^2 \asymp \sum_{j=1}^d \int_{[-\frac{1}{2}, \frac{1}{2}]} \int_{M_\Gamma} \frac{|f(x + ta_j) - f(x)|^2}{|t|^{1+2s}} dx dt. \quad (3.6)$$

The implicit constants in (3.5) and (3.6) depend on  $s, d, \Gamma$ .

The proof of Lemma 3.2 is included in the Appendix.

#### 4. PROOFS OF THE MAIN THEOREMS

This section gives proofs of our main results, Theorems 1.3 and 1.5. We have chosen to organize the proofs into digestible sections of preparatory technical results. In particular, in Section 4.1 we prove a necessary Sobolev embedding for bracket products, and in Section 4.2 we prove a crucial lemma on the rank of  $H^{1/2}$ -valued matrix functions. Finally, in Section 4.3 we combine the preparatory results and prove Theorems 1.3 and 1.5.

**4.1. A Sobolev embedding for bracket products.** Given a lattice  $\Lambda \subset \mathbb{R}^d$  and  $f, g \in L^2(\mathbb{R}^d)$ , it will be convenient to define the *bracket product* of  $g, h$  by

$$[f, g](x) = [f, g]_\Lambda(x) = \sum_{\lambda \in \Lambda} f(x - \lambda) \overline{g(x - \lambda)}.$$

For background on bracket products and their connection to shift-invariant spaces see, for example, [21, 36].

**Lemma 4.1.** *Let  $0 < s < 1$ . If  $g, h \in H^s(\mathbb{R}^d)$  and  $P_\Lambda(g), P_\Lambda(h) \in L^\infty(\mathbb{R}^d/\Lambda)$  then*

$$\|[g, h]\|_{\dot{H}^s(\mathbb{R}^d/\Lambda)}^2 \leq 2C(s, d) \left( \|P_\Lambda(g)\|_{L^\infty(\mathbb{R}^d/\Lambda)} \|h\|_{\dot{H}^s(\mathbb{R}^d)}^2 + \|P_\Lambda(h)\|_{L^\infty(\mathbb{R}^d/\Lambda)} \|g\|_{\dot{H}^s(\mathbb{R}^d)}^2 \right), \quad (4.1)$$

where  $C(s, d)$  is the constant in (3.4). In particular,  $[g, h]_\Lambda \in H^s(\mathbb{R}^d/\Lambda)$ .

*Proof.* Note that  $P_\Lambda(g), P_\Lambda(h) \in L^\infty(\mathbb{R}^d/\Lambda)$  imply that  $[g, h] \in L^\infty(\mathbb{R}^d/\Lambda) \subset L^2(\mathbb{R}^d/\Lambda)$ . So, by Lemma 3.2, it suffices to show that

$$\int_{M_\Lambda} \int_{M_\Lambda} \frac{|[g, h](x + y) - [g, h](x)|^2}{|y|^{d+2s}} dy dx < \infty.$$

We have,

$$\begin{aligned} & \int_{M_\Lambda} \int_{M_\Lambda} \frac{|[g, h](x + y) - [g, h](x)|^2}{|y|^{d+2s}} dy dx \\ & \leq \int_{M_\Lambda} \int_{M_\Lambda} \frac{(\sum_{\lambda \in \Lambda} |g(x + y - \lambda) \overline{h(x + y - \lambda)} - g(x - \lambda) \overline{h(x - \lambda)}|)^2}{|y|^{d+2s}} dy dx \\ & \leq 2 \int_{M_\Lambda} \int_{M_\Lambda} \frac{(\sum_{\lambda \in \Lambda} |g(x + y - \lambda)| |h(x + y - \lambda) - h(x - \lambda)|)^2}{|y|^{d+2s}} dy dx \end{aligned} \quad (4.2)$$

$$+ 2 \int_{M_\Lambda} \int_{M_\Lambda} \frac{(\sum_{\lambda \in \Lambda} |h(x - \lambda)| |g(x + y - \lambda) - g(x - \lambda)|)^2}{|y|^{d+2s}} dy dx. \quad (4.3)$$

Using (3.4), the expression in (4.2) can be bounded as follows,

$$\begin{aligned} & \int_{M_\Lambda} \int_{M_\Lambda} \frac{(\sum_{\lambda \in \Lambda} |g(x + y - \lambda)| |h(x + y - \lambda) - h(x - \lambda)|)^2}{|y|^{d+2s}} dy dx \\ & \leq \int_{M_\Lambda} \int_{M_\Lambda} \frac{\sum_{\lambda \in \Lambda} |g(x + y - \lambda)|^2 \sum_{\lambda \in \Lambda} |h(x + y - \lambda) - h(x - \lambda)|^2}{|y|^{d+2s}} dy dx \\ & \leq \|P_\Lambda(g)\|_{L^\infty(\mathbb{R}^d/\Lambda)} \int_{M_\Lambda} \int_{M_\Lambda} \frac{\sum_{\lambda \in \Lambda} |h(x + y - \lambda) - h(x - \lambda)|^2}{|y|^{d+2s}} dy dx \\ & \leq \|P_\Lambda(g)\|_{L^\infty(\mathbb{R}^d/\Lambda)} \int_{\mathbb{R}^d} \int_{M_\Lambda} \frac{|h(x + y) - h(x)|^2}{|y|^{d+2s}} dy dx \\ & \leq C(d, s) \|P_\Lambda(g)\|_{L^\infty(\mathbb{R}^d/\Lambda)} \|h\|_{\dot{H}^s(\mathbb{R}^d)}^2. \end{aligned}$$

This, together with a similar bound for (4.3), gives (4.1).  $\square$



**4.2. Rank constraints for  $H^{1/2}$ -valued matrices on the torus.** In this section, we prove the following technical lemma which is crucially needed in the proofs of our main theorems.

**Lemma 4.2.** *Let  $\Lambda \subset \mathbb{R}^d$  be a lattice. For almost every  $x$ , let  $P(x)$  be a Hermitian positive semi-definite  $n \times n$  matrix with entries  $p_{i,j} \in H^{1/2}(\mathbb{R}^d/\Lambda)$  for all  $i, j \in \{1, \dots, n\}$ . If, for some  $t > 0$ ,  $P$  satisfies the condition*

$$tP(x) \leq P(x)^2, \quad \text{a.e. } x \in \mathbb{R}^d/\Lambda, \quad (4.4)$$

*then the rank of  $P(x)$  is constant a.e.*

The proof of Lemma 4.2 requires two additional preparatory results, Lemmas 4.3 and 4.4. We first state and prove these preparatory lemmas, and then use them to prove Lemma 4.2 at the end of this section.

**Lemma 4.3.** *Let  $0 < s < 1$ . Let  $\Lambda \subset \mathbb{R}^d$  be a lattice. For almost every  $x$ , let  $P(x)$  be a Hermitian positive semi-definite  $n \times n$  matrix with entries  $p_{i,j} \in H^s(\mathbb{R}^d/\Lambda)$  for all  $i, j \in \{1, \dots, n\}$ . For a given  $x \in \mathbb{R}^d/\Lambda$  such that  $P(x)$  is defined, let  $\lambda_1(x) \geq \dots \geq \lambda_n(x) \geq 0$  denote the eigenvalues of  $P(x)$ . Then, for each  $1 \leq k \leq n$ , the eigenvalue function  $\lambda_k \in H^s(\mathbb{R}^d/\Lambda)$ .*

*Proof.* The Courant-Fischer-Weyl min-max theorem, e.g., Corollary III.1.2 in [16], says that

$$\lambda_k(\xi) = \max\{\min\{\langle u, P(\xi)u \rangle : u \in U, |u| = 1\} : \dim(U) = k\},$$

where the minimum is taken over all  $k$ -dimensional subspaces  $U$  of  $\mathbb{C}^d$ . Then we have

$$\begin{aligned} |\lambda_k(\eta) - \lambda_k(\xi)| &= |\max\{\min\{\langle u, P(\eta)u \rangle : u \in U, |u| = 1\} : \dim(U) = k\} \\ &\quad - \max\{\min\{\langle v, P(\xi)v \rangle : v \in V, |v| = 1\} : \dim(V) = k\}|. \end{aligned}$$

Without loss of generality, assume that  $\lambda_k(\eta) \geq \lambda_k(\xi)$ . Choose a subspace  $U_0$  that realizes the maximum  $\lambda_k(\eta)$ . Then we have

$$\begin{aligned} |\lambda_k(\eta) - \lambda_k(\xi)| &= \min\{\langle u, P(\eta)u \rangle : u \in U_0, |u| = 1\} \\ &\quad - \max\{\min\{\langle v, P(\xi)v \rangle : v \in V, |v| = 1\} : \dim(V) = k\} \\ &\leq \min\{\langle u, P(\eta)u \rangle : u \in U_0, |u| = 1\} - \min\{\langle v, P(\xi)v \rangle : v \in U_0, |v| = 1\}. \end{aligned}$$

Next, choose  $u_0 \in U_0$  with  $\|u_0\| = 1$  such that the minimum in the right term is achieved at  $u_0$ . Then,

$$\begin{aligned} |\lambda_k(\eta) - \lambda_k(\xi)| &\leq \min\{\langle u, P(\eta)u \rangle : u \in U_0, |u| = 1\} - \langle u_0, P(\xi)u_0 \rangle \\ &\leq \langle u_0, P(\eta)u_0 \rangle - \langle u_0, P(\xi)u_0 \rangle \\ &= \langle u_0, (P(\eta) - P(\xi))u_0 \rangle \\ &\leq \|u_0\|^2 \|P(\eta) - P(\xi)\|_{\text{op}} \\ &\leq \|P(\eta) - P(\xi)\|_{\text{Frob}}, \end{aligned}$$

where the last inequality holds since the Frobenius norm of a matrix controls the spectral norm of a matrix. Thus, by (3.5), we have

$$\begin{aligned} \int_{M_\Lambda} \int_{M_\Lambda} \frac{|\lambda_k(x+y) - \lambda_k(x)|^2}{|y|^{d+2s}} dy dx &\leq \int_{M_\Lambda} \int_{M_\Lambda} \frac{\|P(x+y) - P(x)\|_{\text{Frob}}^2}{|y|^{d+2s}} dy dx \\ &= \sum_{i,j=1}^n \int_{M_\Lambda} \int_{M_\Lambda} \frac{|p_{i,j}(x+y) - p_{i,j}(x)|^2}{|y|^{d+2s}} dy dx \\ &= \sum_{i,j=1}^n \|p_{i,j}\|_{\dot{H}^s(\mathbb{R}^d/\Lambda)}^2 < \infty. \end{aligned}$$

□

It is known that if  $f$  is the characteristic function of a measurable set  $S \subset \mathbb{R}^d$  with positive finite Lebesgue measure then  $f \notin H^{1/2}(\mathbb{R}^d)$ , e.g., see [18]. We need the following version of this result for the Sobolev space of periodic functions  $H^{1/2}(\mathbb{R}^d/\Lambda)$  where  $\Lambda \subset \mathbb{R}^d$  is a lattice.

**Lemma 4.4.** *Let  $\Lambda \subset \mathbb{R}^d$  be a lattice. Suppose that  $g \in H^{1/2}(\mathbb{R}^d/\Lambda)$  and there exists  $S \subset M_\Lambda$  such that  $g(x) = 0$  for a.e.  $x \in S$ , and  $g(x) \geq C > 0$  for a.e.  $x \in M_\Lambda \cap S^c$ . Then either  $|S| = 0$  or  $|S| = |M_\Lambda|$ .*

*Proof.* Without loss of generality, we can assume that  $C = 1$ . Let  $\chi_E$  be the  $\Lambda$ -periodic extension of  $\chi_S$  to  $\mathbb{R}^d$ . Notice that for a.e.  $x, y \in \mathbb{R}^d$ ,

$$|g(x+y) - g(x)| \geq |\chi_E(x+y) - \chi_E(x)|,$$

and so, by equation (3.5),  $\chi_E \in H^{1/2}(\mathbb{R}^d/\Lambda)$  since  $g \in H^{1/2}(\mathbb{R}^d/\Lambda)$ . Therefore, it suffices to prove the lemma in the case that  $g = \chi_E$  for some  $E \subset \mathbb{R}^d/\Lambda$ . The proof is divided into two cases depending on whether  $d = 1$  or  $d \geq 2$ .

*Case 1.* We begin by addressing the case  $d = 1$ . In this case  $\Lambda = \alpha\mathbb{Z}$  for some  $\alpha > 0$ , and  $M_\Lambda = [-\alpha/2, \alpha/2)$ . For the sake of contradiction, suppose there exists a set  $S \subset [-\alpha/2, \alpha/2)$  with  $0 < |S| < \alpha$  such that  $g$  is the  $\Lambda$ -periodic extension of  $\chi_S$  to  $\mathbb{R}$ .

For any interval  $I \subset [-\alpha/2, \alpha/2)$ , we have

$$\begin{aligned} \frac{1}{|I|^2} \int_I \int_I |g(x) - g(y)| dx dy &= \frac{1}{|I|^2} \int_I \int_{I-x} |g(x+y) - g(x)| dy dx \\ &\leq \frac{1}{|I|^2} \left( \int_I \int_{I-x} \frac{|g(x+y) - g(x)|^2}{|y|^2} dy dx \right)^{1/2} \left( \int_I \int_{I-x} |y|^2 dy dx \right)^{1/2} \\ &\leq \left( \int_I \int_{I-x} \frac{|g(x+y) - g(x)|^2}{|y|^2} dy dx \right)^{1/2}. \end{aligned} \tag{4.5}$$

Since  $g$  is the indicator function of a set, one has

$$\int_I \int_{\alpha/2 \leq |y| \leq \alpha} \frac{|g(x+y) - g(x)|^2}{|y|^2} dy dx \leq \int_I \int_{\alpha/2 \leq |y| \leq \alpha} \frac{1}{|\alpha/2|^2} dy dx \leq \frac{4|I|}{\alpha}. \tag{4.6}$$

If  $x \in I \subset [-\alpha/2, \alpha/2) = M_\Lambda$  then  $I - x \subset [-\alpha, \alpha]$ . This, together with (4.6), implies that

$$\begin{aligned} \int_I \int_{I-x} \frac{|g(x+y) - g(x)|^2}{|y|^2} dy dx &\leq \int_I \int_{-\alpha}^{\alpha} \frac{|g(x+y) - g(x)|^2}{|y|^2} dy dx \\ &\leq \int_I \int_{-\alpha/2}^{\alpha/2} \frac{|g(x+y) - g(x)|^2}{|y|^2} dy dx + \frac{4|I|}{\alpha}. \end{aligned} \quad (4.7)$$

Using  $g \in H^{1/2}(\mathbb{R}/\Lambda)$ , (3.5), (4.5), (4.7), and absolute continuity of the Lebesgue integral, it follows that

$$\lim_{|I| \rightarrow 0} \frac{1}{|I|^2} \int_I \int_I |g(x) - g(y)| dx dy = 0. \quad (4.8)$$

Since  $0 < |S| < \alpha$ , for every sufficiently small  $\epsilon > 0$ , there exists an interval  $Q_\epsilon \subset [-\alpha/2, \alpha/2)$  such that  $|Q_\epsilon| < \epsilon$  and  $|Q_\epsilon \cap S| = |Q_\epsilon \cap S^c| = |Q_\epsilon|/2$  (for example, this follows from the Lebesgue differentiation theorem). So, for every sufficiently small  $\epsilon > 0$ ,

$$\begin{aligned} \frac{1}{|Q_\epsilon|^2} \int_{Q_\epsilon} \int_{Q_\epsilon} |g(x) - g(y)| dx dy &\geq \frac{1}{|Q_\epsilon|^2} \int_{Q_\epsilon \cap S} \int_{Q_\epsilon \cap S^c} |g(x) - g(y)| dx dy \\ &= \frac{1}{|Q_\epsilon|^2} \int_{Q_\epsilon \cap S} \int_{Q_\epsilon \cap S^c} 1 dx dy \\ &= \frac{|Q_\epsilon \cap S| |Q_\epsilon \cap S^c|}{|Q_\epsilon|^2} = 1/4. \end{aligned} \quad (4.9)$$

On the other hand, by (4.8),

$$\lim_{\epsilon \rightarrow 0} \frac{1}{|Q_\epsilon|^2} \int_{Q_\epsilon} \int_{Q_\epsilon} |g(x) - g(y)| dx dy = 0. \quad (4.10)$$

Since (4.9) and (4.10) form a contradiction, it follows that either  $|S| = 0$  or  $|S| = \alpha = |M_\Lambda|$ .

*Case 2.* Next, we address the case  $d \geq 2$ . Suppose that  $S \subset M_\Lambda$  and that  $g$  is the  $\Lambda$ -periodic extension of  $\chi_S$  to  $\mathbb{R}^d$ . We will show that either  $|S| = 0$  or  $|S| = |M_\Lambda|$ .

Let  $\{a_j\}_{j=1}^d \subset \Lambda$  be a basis for  $\Lambda$ . Recall that  $M_\Lambda = \{\sum_{j=1}^d t_j a_j : -1/2 \leq t_j \leq 1/2\}$ . For each fixed  $x \in M_\Lambda$  and  $1 \leq k \leq d$ , define for  $t \in [-1/2, 1/2)$

$$\psi_{x,k}(t) = g(x + ta_k).$$

Note that for a.e.  $x \in M_\Lambda$ ,  $\psi_{x,k}$  is 1-periodic and  $\psi_{x,k} \in L^2(\mathbb{R}/\mathbb{Z})$ . Also, by (3.6)

$$\begin{aligned} \int_{M_\Lambda} \|\psi_{x,k}\|_{\dot{H}^{1/2}(\mathbb{R}/\mathbb{Z})}^2 dx &\asymp \int_{M_\Lambda} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{|\psi_{x,k}(s+t) - \psi_{x,k}(s)|^2}{|t|^2} ds dt dx \\ &= \int_{M_\Lambda} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{|g(x + (s+t)a_k) - g(x + sa_k)|^2}{|t|^2} ds dt dx \\ &= \int_{M_\Lambda} \int_{-1/2}^{1/2} \frac{|g(y + ta_k) - g(y)|^2}{|t|^2} dt dy < \infty. \end{aligned}$$

Thus, for each  $1 \leq k \leq d$  and a.e.  $x \in M_\Lambda$ , we have  $\psi_{x,k} \in H^{1/2}(\mathbb{R}/\mathbb{Z})$ . However, since  $g(x) \in \{0, 1\}$  a.e., we also have that for each  $1 \leq k \leq d$  and almost every  $x \in M_\Lambda$

$$\psi_{x,k}(t) \in \{0, 1\}, \quad \text{for a.e. } t \in \mathbb{R}. \quad (4.11)$$

It follows from Case 1 that for each  $1 \leq k \leq d$  and almost every  $x \in M_\Lambda$

$$g(x + ta_k) = 0 \text{ for a.e. } t \in \mathbb{R}, \quad \text{or} \quad g(x + ta_k)(t) = 1 \text{ for a.e. } t \in \mathbb{R}. \quad (4.12)$$

To complete the proof it now suffices to show that  $g(x) = g(y)$  for a.e.  $x, y \in M_\Lambda$ . For this, it suffices to show that  $g(\sum_{j=1}^d t_j a_j) = g(\sum_{j=1}^d s_j a_j)$  for a.e.  $t = (t_1, \dots, t_d) \in [-1/2, 1/2]^d$  and  $s = (s_1, \dots, s_d) \in [-1/2, 1/2]^d$ . Similarly to Lemma 2 in [20], and using (4.12), one has

$$\begin{aligned} & \int_{[-1/2, 1/2]^d} \int_{[-1/2, 1/2]^d} |g(\sum_{j=1}^d t_j a_j) - g(\sum_{j=1}^d s_j a_j)| dt ds \\ & \leq \int_{[-1/2, 1/2]^d} \int_{[-1/2, 1/2]^d} |g(\sum_{j=1}^d t_j a_j) - g(s_1 a_1 + \sum_{j=2}^d t_j a_j)| ds dt \\ & + \int_{[-1/2, 1/2]^d} \int_{[-1/2, 1/2]^d} |g(s_1 a_1 + \sum_{j=2}^d t_j a_j) - g(\sum_{j=1}^2 s_j a_j + \sum_{j=3}^d t_j a_j)| ds dt \\ & \vdots \\ & + \int_{[-1/2, 1/2]^d} \int_{[-1/2, 1/2]^d} |g(\sum_{j=1}^{d-1} s_j a_j + t_d a_d) - g(\sum_{j=1}^d s_j a_j)| ds dt \\ & = 0. \end{aligned}$$

Thus,  $g(x) = g(y)$  for almost every  $x, y \in M_\Lambda$ .  $\square$

For perspective, the hypothesis of Lemma 4.4 implies the condition  $C|g(x)| \leq |g(x)|^2$  for a.e.  $x \in \mathbb{R}^d$ , which may be viewed as a scalar version of the matrix-valued hypothesis (4.4). In particular, Lemma 4.2 may be thought of as a matrix-valued generalization of Lemma 4.4. We are now ready to prove Lemma 4.2.

*Proof of Lemma 4.2.* Lemma 4.3 shows that the eigenvalue functions  $\lambda_k$  of  $P$  are in  $H^{1/2}(\mathbb{R}^d/\Lambda)$ . Condition (4.4) implies that for almost every  $x \in \mathbb{R}^d/\Lambda$ ,  $\lambda_k(x) = 0$  or  $\lambda_k(x) \geq t > 0$ . From Lemma 4.4 we have that  $\lambda_k$  is either zero almost everywhere or positive almost everywhere. Therefore, the rank of  $P(x)$  is constant almost everywhere.  $\square$

**4.3. Combining everything: proofs of the main theorems.** In this section, we combine all of our preparatory results and prove Theorems 1.3 and 1.5.

*Proof of Theorem 1.3.* Assume, for the sake of contradiction, that  $\widehat{f}_k \in H^{1/2}(\mathbb{R}^d)$  for all  $1 \leq k \leq K$ .

*Step I.* Note that the periodizations  $P_{\Lambda^*}(\widehat{f}_k) \in L^\infty(\mathbb{R}^d/\Lambda^*)$  and  $P_{\Gamma^*}(\widehat{f}_k) \in L^\infty(\mathbb{R}^d/\Gamma^*)$  for each  $1 \leq k \leq K$ . To see this, let  $e_k$  be the  $k$ th canonical basis vector for  $\mathbb{R}^K$ , and use (2.2) to obtain

$$\left([\widehat{f}_k, \widehat{f}_k]_{\Lambda^*}\right)^2 \leq \sum_{m=1}^K \left|[\widehat{f}_m, \widehat{f}_k]_{\Lambda^*}\right|^2 = \langle P e_k, P e_k \rangle = \langle P^2 e_k, e_k \rangle \leq \langle t P e_k, e_k \rangle = t [\widehat{f}_k, \widehat{f}_k]_{\Lambda^*}.$$

Recalling that  $P_{\Lambda^*}(\widehat{f}_k) = [\widehat{f}_k, \widehat{f}_k]_{\Lambda^*}$ , it follows that  $|P_{\Lambda^*}(\widehat{f}_k)(x)| \leq t$  for a.e.  $x \in \mathbb{R}^d$ , so that  $P_{\Lambda^*}(\widehat{f}_k) \in L^\infty(\mathbb{R}^d/\Lambda^*)$ . Similar reasoning, together with Theorem 2.3, shows that  $P_{\Gamma^*}(\widehat{f}_k) \in L^\infty(\mathbb{R}^d/\Gamma^*)$ .

*Step II.* Lemma 4.1 implies that the bracket products satisfy  $[\widehat{f}_m, \widehat{f}_n]_{\Lambda^*} \in H^{1/2}(\mathbb{R}^d/\Lambda^*)$  and  $[\widehat{f}_m, \widehat{f}_n]_{\Gamma^*} \in H^{1/2}(\mathbb{R}^d/\Gamma^*)$  for all  $1 \leq m, n \leq K$ . So, all entries of the Gramian  $P_{\Lambda^*}(\widehat{F})$  are in  $H^{1/2}(\mathbb{R}^d/\Lambda^*)$ , and all entries of  $P_{\Gamma^*}(\widehat{F})$  are in  $H^{1/2}(\mathbb{R}^d/\Gamma^*)$ . Combining Lemma 4.2, Theorem 2.3, and equation (2.2) shows that the rank of both  $P_{\Lambda^*}(\widehat{F})(x)$  and  $P_{\Gamma^*}(\widehat{F})(x)$  are constant almost everywhere.

*Step III.* By Proposition 2.1,  $\text{rank}[P_{\Gamma^*}(\widehat{F})(x)] = \rho(F, \Gamma)$  a.e., and  $\text{rank}[P_{\Lambda^*}(\widehat{F})(x)] = \rho(F, \Lambda)$  a.e. Then, from Theorem 2.2, we have

$$\rho(F, \Lambda) = \text{rank}[P_{\Lambda^*}(\widehat{F})(x)] = \sum_{k \in R} \text{rank}[P_{\Gamma^*}(\widehat{F})(x + k)] = \rho(F, \Gamma) \text{card}(R) = \rho(F, \Gamma)[\Gamma : \Lambda],$$

where  $R$  is a set of representatives of the quotient  $\Lambda^*/\Gamma^*$ . Since this contradicts the assumption that  $[\Gamma : \Lambda]$  is not a divisor of  $\rho(F, \Lambda)$ , we must have that  $\widehat{f}_k \notin H^{1/2}(\mathbb{R}^d)$  for some  $1 \leq k \leq K$ .  $\square$

*Proof of Theorem 1.5.* Assume, for the sake of contradiction, that  $\widehat{f}_k \in H^{1/2}(\mathbb{R}^d)$  for all  $1 \leq k \leq K$ . Using Lemma 4.1, similar reasoning as in the proof of Theorem 1.3 implies that each entry of  $P = P_{\Lambda^*}(\widehat{F})$  is in  $H^{1/2}(\mathbb{R}^d/\Lambda^*)$ . The assumption that  $K = \rho(F, \Lambda)$ , along with Lemma 4.2 and (2.2), implies that  $P$  is full rank almost everywhere. This forces the eigenvalue functions,  $\lambda_k$ , of  $P$  to be nonzero almost everywhere for all  $1 \leq k \leq K = \rho(F, \Lambda)$ . However, equation (2.2) shows that the eigenvalue functions are then bounded below by  $t^{-1}$  almost everywhere. This is equivalent to  $P$  satisfying the lower bound in equation (2.1). The upper bound also follows from (2.2). Thus,  $\mathcal{T}^\Lambda(F)$  forms a Riesz basis for  $V^\Lambda(F)$  which gives a contradiction.  $\square$

## 5. EXAMPLES

The first two examples in this section show that there are multiply generated shift-invariant spaces for which the hypotheses of Theorem 1.3 hold, but for which the conclusion of the theorem only holds for a single generator. The collection of smooth compactly supported functions on  $\mathbb{R}^d$  will be denoted by  $C_c^\infty(\mathbb{R}^d)$ .

**Example 5.1.** Let  $I = [-1/2, 1/2]^d$ . Define  $f_1 \in L^2(\mathbb{R}^d)$  by  $\widehat{f}_1 = \chi_I$ . Take any  $g \in C_c^\infty(\mathbb{R}^d)$  that is supported on  $I$  and satisfies  $\|g\|_2 = 1$ , and define  $f_2 \in L^2(\mathbb{R}^d)$  by  $\widehat{f}_2 = g$ .

Let  $F = \{f_1, f_2\}$ ,  $\Lambda = \mathbb{Z}^d$ , and  $\Gamma = (\frac{1}{2}\mathbb{Z}) \times \mathbb{Z}^{d-1}$ . The space  $V^\Lambda(F) = V^{\mathbb{Z}^d}(f_1, f_2)$  has the following properties:

- $\mathcal{T}^\Lambda(F)$  is a frame for  $V^\Lambda(F)$ ;
- $V^\Lambda(F)$  is  $\Gamma$ -invariant (it is actually translation invariant);
- $\widehat{f}_1 \notin H^{1/2}(\mathbb{R}^d)$  and  $\widehat{f}_2 \in C_c^\infty(\mathbb{R}^d) \subset H^{1/2}(\mathbb{R}^d)$ ;
- $\rho(F, \Lambda) = 1$  and  $[\Gamma : \Lambda] = 2$ , so that  $[\Gamma : \Lambda]$  does not divide  $\rho(F, \Lambda)$ .

This can be verified by computing the Gramian  $P_{\Lambda^*}(\widehat{F})(x)$ . Note that  $\Lambda^* = \Lambda = \mathbb{Z}^d$ . Since  $P_{\Lambda^*}(\widehat{F})(x)$  is  $\Lambda^*$ -periodic, it suffices to only consider  $x \in I$  in the subsequent discussion. A computation shows that for  $x \in I$

$$P_{\Lambda^*}(\widehat{F})(x) = P_{\mathbb{Z}^d}(\widehat{F})(x) = \begin{pmatrix} [\widehat{f}_1, \widehat{f}_1]_{\mathbb{Z}^d}(x) & [\widehat{f}_1, \widehat{f}_2]_{\mathbb{Z}^d}(x) \\ [\widehat{f}_2, \widehat{f}_1]_{\mathbb{Z}^d}(x) & [\widehat{f}_2, \widehat{f}_2]_{\mathbb{Z}^d}(x) \end{pmatrix} = \begin{pmatrix} 1 & \overline{g(x)} \\ g(x) & |g(x)|^2 \end{pmatrix}.$$

A further computation shows that

$$\left(P_{\Lambda^*}(\widehat{F})(x)\right)^2 = (1 + |g(x)|^2) \begin{pmatrix} 1 & \overline{g(x)} \\ g(x) & |g(x)|^2 \end{pmatrix} = (1 + |g(x)|^2) P_{\Lambda^*}(\widehat{F})(x)$$

Since  $g \in C_c^\infty(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ , we have the operator inequality

$$P_{\Lambda^*}(\widehat{F})(x) \leq \left(P_{\Lambda^*}(\widehat{F})(x)\right)^2 \leq (1 + \|g\|_\infty^2) P_{\Lambda^*}(\widehat{F})(x).$$

So, by (2.2),  $\mathcal{T}^\Lambda(F)$  is a frame for  $V^\Lambda(F)$ .

The remaining properties can also be checked easily. Similar computations as above, together with Theorem 2.2, show that  $V^\Lambda(F)$  is  $\Gamma$ -invariant. A direct computation shows that  $\widehat{f}_1 \notin H^{1/2}(\mathbb{R}^d)$ . The condition  $\rho(F, \Lambda) = 1$  can be seen by using Proposition 2.1 and noting that  $P_{\Lambda^*}(\widehat{F})(x)$  has rank 1 for all  $x \in I$ . Finally, it is easily verified that  $[\Lambda : \Gamma] = 2$ .

**Example 5.2.** Fix any integer  $N \geq 2$ . Let  $I = [-1/2, 1/2)$  and define  $f_{N+1} \in L^2(\mathbb{R})$  by  $\widehat{f_{N+1}} = \chi_I$ . Fix  $0 < \epsilon < \frac{1}{2N}$ . Select  $f \in C_c^\infty(\mathbb{R})$  with  $\|f\|_2 = 1$  such that  $f$  is supported on  $[0, 1/N]$ , and such that  $|\widehat{f}(x)| \leq \epsilon$  for all  $x \in I$ . For example, such an  $f$  can be constructed by suitably dilating and translating a given smooth compactly supported function. For  $1 \leq n \leq N$ , define  $f_n(x) = f(x - n/N)$ .

Define  $F = \{f_n\}_{n=1}^{N+1} \subset L^2(\mathbb{R})$ ,  $\Lambda = \mathbb{Z}$ , and  $\Gamma = \frac{1}{N}\mathbb{Z}$ . The space  $V^\Lambda(F)$  satisfies the following properties

- $\mathcal{T}^\Lambda(F)$  is a Riesz basis for  $V^\Lambda(F)$ ;
- $V^\Lambda(F)$  is invariant under  $\Gamma$ ;
- $\rho(F, \Lambda) = N + 1$  and  $[\Gamma : \Lambda] = N$ , so that  $[\Gamma : \Lambda]$  does not divide  $\rho(F, \Lambda)$ ;
- $\widehat{f_n} \in H^{1/2}(\mathbb{R})$  for each  $1 \leq n \leq N$ ,
- $\widehat{f_{N+1}} \notin H^{1/2}(\mathbb{R})$ .

The singly generated system  $V^\mathbb{Z}(f_{N+1})$  is easily seen to be  $\frac{1}{N}\mathbb{Z}$ -invariant by Theorem 2.2 (in fact,  $V^\Lambda(f_{N+1})$  is translation invariant). Moreover, the space  $V^\mathbb{Z}(f_1, \dots, f_N)$  is  $\frac{1}{N}\mathbb{Z}$ -invariant by construction. It follows that  $V^\Lambda(F) = V^\mathbb{Z}(f_1, \dots, f_{N+1})$  is  $\frac{1}{N}\mathbb{Z}$ -invariant. Also,  $\widehat{f_n} \in H^{1/2}(\mathbb{R})$  for each  $1 \leq n \leq N$  since  $f_n \in C_c^\infty(\mathbb{R})$ .

Since  $\mathcal{T}^{\mathbb{Z}}(\{f_n\}_{n=1}^N)$  is an orthonormal basis for  $V^{\mathbb{Z}}(\{f_n\}_{n=1}^N)$ , one has for  $1 \leq j, k \leq N$  that  $[\widehat{f_j}, \widehat{f_k}]_{\mathbb{Z}}(x) = \delta_{j,k}$  for a.e.  $x \in \mathbb{R}$ , for example, see [36]. Also, if  $1 \leq n \leq N$ , then for  $x \in I$ ,

$$[\widehat{f_n}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) = \sum_{j \in \mathbb{Z}} \widehat{f_n}(x-j) \chi_I(x-j) = \widehat{f_n}(x) = e^{-2\pi i n x / N} \widehat{f}(x).$$

By our assumptions on  $f$ , we have that for  $1 \leq n \leq N$ , and  $x \in I$ ,

$$\left| [\widehat{f_{N+1}}, \widehat{f_n}]_{\mathbb{Z}}(x) \right| = \left| [\widehat{f_n}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \right| = |\widehat{f}(x)| \leq \epsilon. \quad (5.1)$$

Recalling that  $P_{\mathbb{Z}}(\widehat{F})(x)$  is  $\mathbb{Z}$ -periodic, we have that for all  $x \in I$ ,

$$P_{\mathbb{Z}}(\widehat{F})(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & [\widehat{f_1}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \\ 0 & 1 & \cdots & 0 & [\widehat{f_2}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & [\widehat{f_N}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \\ [\widehat{f_{N+1}}, \widehat{f_1}]_{\mathbb{Z}}(x) & [\widehat{f_{N+1}}, \widehat{f_2}]_{\mathbb{Z}}(x) & \cdots & [\widehat{f_{N+1}}, \widehat{f_N}]_{\mathbb{Z}}(x) & 1 \end{pmatrix}.$$

The Gershgorin circle theorem, together with (5.1), shows that all eigenvalues of  $P_{\mathbb{Z}}(\widehat{F})(x)$  lie in the interval  $[1 - N\epsilon, 1 + N\epsilon]$ . Since  $0 < \epsilon < \frac{1}{2N}$ , the condition (2.1) holds with  $t = 2$ , and hence  $\mathcal{T}^{\Lambda}(F)$  is a Riesz basis for  $V^{\Lambda}(F)$ . Moreover, since  $P_{\mathbb{Z}}(\widehat{F})(x)$  is full rank for a.e.  $x \in I$ , Proposition 2.1 shows that  $\rho(F, \Lambda) = N + 1$ .

The next example shows that there are multiply generated shift-invariant spaces for which the hypotheses of Theorem 1.5 hold and for which the conclusion only holds for a single generator.

**Example 5.3.** Let  $J = [-1/4, 1/4]$ . Define  $f_1 \in L^2(\mathbb{R})$  by  $\widehat{f_1} = \chi_J$ . Select  $f_2 \in C_c^{\infty}(\mathbb{R})$  such that  $f_2$  is supported in  $[-1/2, 1/2]$ ,  $\|f_2\|_2 = 1$ , and  $|\widehat{f_2}(x)| < 1/2$  for all  $x \in J$ .

Define  $F = \{f_1, f_2\}$  and  $\Lambda = \mathbb{Z}$ . The space  $V^{\Lambda}(F)$  satisfies the following properties:

- $\mathcal{T}^{\Lambda}(F)$  is a frame, but not a Riesz basis, for  $V^{\Lambda}(F)$ ;
- The minimal number of generators  $\rho(F, \Lambda) = 2$ ;
- $\widehat{f_1} \notin H^{1/2}(\mathbb{R})$  and  $\widehat{f_2} \in H^{1/2}(\mathbb{R})$ .

Recall that  $P_{\mathbb{Z}}(\widehat{F})(x)$  is  $\mathbb{Z}$ -periodic. A computation shows that for  $x \in [-1/2, 1/2]$

$$P_{\mathbb{Z}}(\widehat{F})(x) = \begin{pmatrix} \chi_J(x) & \chi_J(x) \widehat{f_2}(x) \\ \chi_J(x) \widehat{f_2}(x) & 1 \end{pmatrix}.$$

For  $1/4 < |x| < 1/2$ , we have

$$P_{\mathbb{Z}}(\widehat{F})(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so that  $\lambda_1(x) = 1$  and  $\lambda_2(x) = 0$ , and for  $|x| < 1/4$ , we have

$$P_{\mathbb{Z}}(\widehat{F})(x) = \begin{pmatrix} 1 & \widehat{f_2}(x) \\ \widehat{f_2}(x) & 1 \end{pmatrix}$$



so that  $\lambda_1(x) = 1 + |\widehat{f}_2(x)|$  and  $\lambda_2(x) = 1 - |\widehat{f}_2(x)|$ .

By (2.1),  $\mathcal{T}^{\mathbb{Z}}(F)$  is not a Riesz basis for  $V^{\mathbb{Z}}(F)$ . However Proposition 2.1, (2.2), and  $|\widehat{f}_2(x)| < 1/2$  for  $x \in J$ , show that  $\mathcal{T}^{\mathbb{Z}}(F)$  is a frame for  $V^{\mathbb{Z}}(F)$  and  $\rho(F, \mathbb{Z}) = 2$ .

#### ACKNOWLEDGMENTS

The authors thank Jeff Hogan, Joe Lakey and Qiyu Sun for helpful discussions related to the literature on shift-invariant invariant spaces. The authors also thank Charly Gröchenig for useful comments related to the Balian-Low theorem.

D. Hardin was partially supported by NSF DMS Grant 1109266, NSF DMS Grant 0934630, NSF DMS Grant 1521749, NSF DMS Grant 1516400, and NSF DMS Grant 1412428. M. Northington was partially supported by NSF DMS Grant 0934630, NSF DMS Grant 1521749, and NSF DMS Grant 1211687. A. Powell was partially supported by NSF DMS Grant 1211687 and NSF DMS Grant 1521749. A. Powell gratefully acknowledges the hospitality and support of the Academia Sinica Institute of Mathematics (Taipei, Taiwan).

#### APPENDIX

In this Appendix, we include a proof of the characterization of Sobolev spaces on the torus from Lemma 3.2.

*Proof of Lemma 3.2.*

*Step I.* We first prove (3.5). By Parseval's theorem

$$\begin{aligned} |M_{\Gamma}| \int_{M_{\Gamma}} \int_{M_{\Gamma}} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} dx dy &= \int_{M_{\Gamma}} \frac{\sum_{\xi \in \Gamma^*} |\widehat{f}(\xi)|^2 |e^{-2\pi i \xi \cdot y} - 1|^2}{|y|^{d+2s}} dy \\ &= \sum_{\xi \in \Gamma^*} |\widehat{f}(\xi)|^2 G(\xi), \end{aligned} \quad (5.2)$$

where  $G(\xi) = \int_{M_{\Gamma}} \frac{|e^{-2\pi i \xi \cdot y} - 1|^2}{|y|^{d+2s}} dy$ . For  $\xi \neq 0$ , let  $\xi' = \xi/|\xi|$  and note that

$$G(\xi) = 2 \int_{M_{\Gamma}} \frac{1 - \cos(2\pi \xi \cdot y)}{|y|^{d+2s}} dy = 2|\xi|^{2s} \int_{|\xi| M_{\Gamma}} \frac{1 - \cos(2\pi \xi' \cdot y)}{|y|^{d+2s}} dy. \quad (5.3)$$

Since  $\Gamma$  and  $\Gamma^*$  are full-rank lattices, there exists  $C > 0$  such that  $|\xi| \geq C > 0$  for all  $\xi \in \Gamma^* \setminus \{0\}$ . Moreover, there exists  $r > 0$  such that  $B(0, r) = \{x \in \mathbb{R}^d : |x| < r\} \subset M_{\Gamma}$ . Thus,

$$\forall \xi \in \Gamma^* \setminus \{0\}, \quad B(0, Cr) \subset |\xi| M_{\Gamma}. \quad (5.4)$$

Combining (5.3) and (5.4) shows that  $G(\xi)$  satisfies the lower bound

$$\forall \xi \in \Gamma^* \setminus \{0\}, \quad 2|\xi|^{2s} \int_{B(0, Cr)} \frac{1 - \cos(2\pi \xi' \cdot y)}{|y|^{d+2s}} dy \leq G(\xi). \quad (5.5)$$

Moreover (5.3) gives the upper bound

$$G(\xi) \leq 2|\xi|^{2s} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi \xi' \cdot y)}{|y|^{d+2s}} dy. \quad (5.6)$$

Notice that the integrals in (5.5) and (5.6) are invariant under  $y \mapsto Uy$  for all unitary  $U : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . It follows that these integrals are independent of the actual value of  $\xi'$ . The integral in (5.5) is positive since the integrand is a.e. positive. It is also easily seen that the integral in (5.6) is finite. Thus,

$$G(\xi) \asymp |\xi|^{2s}.$$

This, together with (5.2) completes the proof of (3.5).

*Step II.* Next, we prove (3.6). By Parseval's theorem

$$\begin{aligned} |M_\Gamma| \sum_{j=1}^d \int_{[-\frac{1}{2}, \frac{1}{2})} \int_{M_\Gamma} \frac{|f(x + ta_j) - f(x)|^2}{|t|^{1+2s}} dx dt &= \sum_{j=1}^d \int_{[-\frac{1}{2}, \frac{1}{2})} \frac{\sum_{\xi \in \Gamma^*} |\widehat{f}(\xi)|^2 |e^{-2\pi i \xi \cdot ta_j} - 1|^2}{|t|^{1+2s}} dt \\ &= \sum_{\xi \in \Gamma^*} |\widehat{f}(\xi)|^2 H(\xi), \end{aligned} \quad (5.7)$$

where

$$H(\xi) = \sum_{j=1}^d \int_{[-\frac{1}{2}, \frac{1}{2})} \frac{|e^{-2\pi i t \xi \cdot a_j} - 1|^2}{|t|^{1+2s}} dt.$$

For  $\xi \in \Gamma^* \setminus \{0\}$ , let  $\xi' = \xi/|\xi|$ , and note that

$$\begin{aligned} H(\xi) &= 2 \sum_{j=1}^d \int_{[-\frac{1}{2}, \frac{1}{2})} \frac{1 - \cos(2\pi t \xi \cdot a_j)}{|t|^{1+2s}} dt \\ &= 2|\xi|^{2s} \sum_{j=1}^d \int_{-|\xi|/2}^{|\xi|/2} \frac{1 - \cos(2\pi t \xi' \cdot a_j)}{|t|^{1+2s}} dt \end{aligned} \quad (5.8)$$

$$\leq 2d|\xi|^{2s} \int_{-\infty}^{\infty} \frac{1}{|t|^{1+2s}} dt. \quad (5.9)$$

Let  $A$  be the  $d \times d$  matrix with  $a_j$  as its  $j$ th column. Then  $a_j = Ae_j$  where  $\{e_j\}_{j=1}^d \subset \mathbb{Z}^d$  is the canonical basis for  $\mathbb{R}^d$ . Since  $\{a_j\}_{j=1}^d$  is a basis for  $\Gamma$ , the matrix  $A$  is invertible. If  $\sigma_d \geq \sigma_1 > 0$  are the respective largest and smallest singular values of  $A$  (and hence also  $A^*$ ) then

$$\forall x \in \mathbb{R}^d, \quad \sigma_1 |x| \leq |A^* x| \leq \sigma_d |x|.$$

Moreover, if  $\xi \in \Gamma \setminus \{0\}$ , then  $\xi = (A^*)^{-1}z$  for some  $z \in \mathbb{Z}^d \setminus \{0\}$ , so that

$$\forall \xi \in \Gamma^* \setminus \{0\}, \quad \sigma_d^{-1} \leq \sigma_d^{-1} |z| \leq |(A^*)^{-1}z| = |\xi|$$

Let  $\rho = 1/(4\sigma_d)$ . Using that  $(1 - \cos(2\pi\theta)) \geq \theta^2/2$  for all  $|\theta| \leq 1/4$ , it follows from (5.8) that for all  $\xi \in \Gamma^* \setminus \{0\}$

$$\begin{aligned}
H(\xi) &= 2|\xi|^{2s} \sum_{j=1}^d \int_{-|\xi|/2}^{|\xi|/2} \frac{1 - \cos(2\pi t A^*(\xi') \cdot e_j)}{|t|^{1+2s}} dt \\
&\geq 2|\xi|^{2s} \sum_{j=1}^d \int_{-\rho}^{\rho} \frac{1 - \cos(2\pi t A^*(\xi') \cdot e_j)}{|t|^{1+2s}} dt \\
&\geq 2|\xi|^{2s} \sum_{j=1}^d \int_{-\rho}^{\rho} \frac{|2\pi t A^*(\xi') \cdot e_j|^2}{2|t|^{1+2s}} dt \\
&= 4\pi^2 |\xi|^{2s} \left( \sum_{j=1}^d |A^*(\xi') \cdot e_j|^2 \right) \int_{-\rho}^{\rho} \frac{1}{|t|^{2s-1}} dt \\
&= 4\pi^2 |\xi|^{2s} |A^*(\xi')|^2 \int_{-\rho}^{\rho} \frac{1}{|t|^{2s-1}} dt \\
&\geq 4\pi^2 \sigma_1 |\xi|^{2s} \int_{-\rho}^{\rho} \frac{1}{|t|^{2s-1}} dt.
\end{aligned} \tag{5.10}$$

Hence, by (5.9) and (5.10)

$$H(\xi) \asymp |\xi|^{2s}.$$

This, together with (5.7) completes the proof of (3.6).  $\square$

## REFERENCES

- [1] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson and U. Molter, Invariance of a shift-invariant space, *Journal of Fourier Analysis and Applications* 16 (2010), no.1, 60–75.
- [2] A. Aldroubi, Q. Sun and H. Wang, Uncertainty principles and Balian-Low type theorems in principal shift-invariant spaces, *Applied and Computational Harmonic Analysis*, 30 (2011), no. 3, 337–347.
- [3] M. Anastasio, C. Cabrelli and V. Paternostro, Invariance of a shift-invariant space in several variables, *Complex Analysis and Operator Theory*, 5 (2011), no. 4, 1031–1050.
- [4] G. Ascensi, H. Feichtinger and N. Kaiblinger, Dilation of the Weyl symbol and Balian-Low theorem, *Transactions of the AMS*, 366 (2014), no. 7, 3865–3880.
- [5] R. Balan and I. Daubechies, Optimal stochastic encoding and approximation schemes using Weyl-Heisenberg sets, *Advances in Gabor analysis, Applied and Numerical Harmonic Analysis*, 259–320, Birkhäuser, Boston, MA, 2003.
- [6] R. Balian, Un principe d'incertitude fort en théorie du signal ou en mécanique quantique, *Comptes Rendus de l'Académie des Sciences*, 292 (1981), no. 20, 1357–1362.
- [7] A. Bastys, Translation invariance of orthogonal multiresolution analyses of  $L^2(\mathbb{R})$ , *Applied and Computational Harmonic Analysis*, 9 (2000), 128–145.
- [8] G. Battle, Heisenberg proof of the Balian-Low theorem, *Letters in Mathematical Physics*, 15 (1988), 175–177.
- [9] J.J. Benedetto, W. Czaja, P. Gadziński, and A.M. Powell, The Balian-Low Theorem and regularity of Gabor systems, *Journal of Geometric Analysis*, 13 (2003), no. 2, 239–254.
- [10] J.J. Benedetto, W. Czaja, and A. Maltsev, The Balian-Low theorem for the symplectic form on  $\mathbb{R}^{2d}$ , *Journal of Mathematical Physics*, 44 (2003), no. 4, 1735–1750.
- [11] J.J. Benedetto, W. Czaja, A.M. Powell, An optimal example for the Balian-low uncertainty principle, *SIAM Journal of Mathematical Analysis*, 38 (2006), no. 1, 333–345.

- [12] J.J. Benedetto, W. Czaja, A.M. Powell, and J. Sterbenz, An endpoint  $(1, \infty)$  Balian–Low Theorem, *Mathematical Research Letters*, 13 (2006), no. 2–4, 467–474.
- [13] J.J. Benedetto, C. Heil, and D. Walnut, Differentiation and the Balian–Low theorem, *Journal of Fourier Analysis and Applications*, 1 (1995), no. 4, 355–402.
- [14] J.J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, *Applied and Computational Harmonic Analysis*, 5 (1998), no. 4, 389–427.
- [15] A. Benyi and T. Oh, The Sobolev inequality on the torus revisited, *Publicationes Mathematicae Debrecen*, 83.3 (2013), 359–374.
- [16] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
- [17] J. Bourgain, Remark on the uncertainty principle for Hilbertian basis, *Journal of Functional Analysis*, 79 (1988), no. 1, 136–143.
- [18] J. Bourgain, H. Brezis and P. Mironescu, Lifting in Sobolev spaces, *Journal d’Analyse Mathématique*, 80 (2000), 37–86.
- [19] M. Bownik, The structure of shift-invariant subspaces of  $L^2(\mathbb{R}^n)$ , *Journal of Functional Analysis*, 177 (2000) 282–309.
- [20] H. Brezis, Y. Li, P. Mironescu and L. Nirenberg, Degree and Sobolev spaces, *Topological Methods in Nonlinear Analysis*, 13 (1999), 181–190.
- [21] P. Casazza and M. Lammers, Bracket products for Weyl–Heisenberg frames, in “Advances in Gabor Analysis”, *Applied and Numerical Harmonic Analysis*, Birkhäuser, Boston, 2003, pages 71–98.
- [22] O. Christensen, *An Introduction to frames and Riesz bases*, *Applied and Numerical Harmonic Analysis*, Birkhäuser, Boston, 2003.
- [23] W. Czaja and A.M. Powell, Recent developments in the Balian–Low theorem, in “Harmonic Analysis and Applications”, *Applied and Numerical Harmonic Analysis*, Birkhäuser, Boston, 2006, pages 79–100.
- [24] I. Daubechies and A.J.E.M. Janssen, Two theorems on lattice expansions, *IEEE Transactions on Information Theory*, 39 (1993), no. 1, 3–6.
- [25] C. de Boor, R.A. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in  $L_2(\mathbb{R}^d)$ , *Journal of Functional Analysis*, 119 (1994), no. 1, 37–78.
- [26] G.B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey, *Journal of Fourier Analysis and Applications*, 3 (1997), no. 3, 207–238.
- [27] J.-P. Gabardo and D. Han, Balian–Low phenomenon for subspace Gabor frames, *Journal of Mathematical Physics*, 45 (2004), no. 8, 3362–3378.
- [28] S.Z. Gautam, A critical exponent Balian–Low theorem, *Mathematical Research Letters*, 15 (2008), no. 3, 471–483.
- [29] K. Gröchenig, personal communication.
- [30] K. Gröchenig and E. Malinnikova, Phase space localization of Riesz bases for  $L^2(\mathbb{R}^d)$ , *Revista Matemática Iberoamericana*, 29 (2013), no. 1, 115–134.
- [31] K. Gröchenig, D. Han, C. Heil, G. Kutyniok, The Balian–Low theorem for symplectic lattices in higher dimensions, *Applied and Computational Harmonic Analysis*, 13 (2002), no. 2, 169–176.
- [32] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Vol 28, Springer-Verlag, Berlin, 1994.
- [33] C. Heil, History and evolution of the density theorem for Gabor frames, *Journal of Fourier Analysis and Applications*, 13 (2007), no. 2, 113–166.
- [34] C. Heil and A.M. Powell, Gabor Schauder bases and the Balian–Low theorem, *Journal of Mathematical Physics*, 47 (2006), no. 11, 113506, 21 pp.
- [35] C. Heil and A.M. Powell, Regularity for complete and minimal Gabor systems on a lattice, *Illinois Journal of Mathematics*, 53 (2009), no. 4, 1077–1094.
- [36] E. Hernández, H. Šikić, G. Weiss and E. Wilson, On the properties of the integer translates of a square integrable function, *Contemporary Mathematics*, 505 (2010), 233–249.
- [37] J. Hogan and J. Lakey, Non-translation-invariance and the synchronization problem in wavelet sampling, *Acta Applicandae Mathematicae*, 107 (2009), 373–398.

- [38] J. Hogan and J. Lakey, Non-translation-invariance in principal shift-invariant spaces, *Advances in analysis*, 471–485, World Scientific Publishing, Hackensack, NJ, 2005.
- [39] A.J.E.M. Janssen, A decay result for certain windows generating orthogonal Gabor bases, *Journal of Fourier Analysis and Applications*, 14 (2008), 1–15.
- [40] M. Lammers, The finite Balian-Low conjecture, *Proceedings of the 11th International Conference on Sampling Theory and Applications 2015* (Washington DC, May 25–29, 2015), 4 pages.
- [41] F. Low, Complete sets of wave packets, in: “A Passion for Physics—Essays in Honor of Geoffrey Chew,” C. DeTar et al., eds., World Scientific, Singapore, 1985, pages 17–22.
- [42] S. Nitzan and J.-F. Olsen, From exact systems to Riesz bases in the Balian-Low theorem, *Journal of Fourier Analysis and Applications*, 17 (2011), no. 4, 567–603.
- [43] S. Nitzan and J.-F. Olsen, A quantitative Balian-Low theorem, *Journal of Fourier Analysis and Applications*, 19 (2013), no. 5, 1078–1092.
- [44] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, NJ, 1970.
- [45] R. Tessera and H. Wang, Uncertainty principles in finitely generated shift-invariant spaces with additional invariance, *Journal of Mathematical Analysis and Applications*, 410 (2014), 134–143.

VANDERBILT UNIVERSITY, DEPARTMENT OF MATHEMATICS, NASHVILLE, TN 37240  
*E-mail address:* `doug.hardin@vanderbilt.edu`

VANDERBILT UNIVERSITY, DEPARTMENT OF MATHEMATICS, NASHVILLE, TN 37240  
*E-mail address:* `michael.c.northington.v@vanderbilt.edu`

VANDERBILT UNIVERSITY, DEPARTMENT OF MATHEMATICS, NASHVILLE, TN 37240  
*E-mail address:* `alexander.m.powell@vanderbilt.edu`